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# Application of Finite Groups to Neutrino Mass Matrices

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## **Abstract**

Recent progress in the application of finite groups to neutrino mass matrices is reviewed, with special emphasis on the tetrahedral symmetry  $A_4$ .

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# 1 Introduction

Using present data from neutrino oscillations, the  $3 \times 3$  neutrino mixing matrix is largely determined, together with the two mass-squared differences [1]. In the Standard Model of particle interactions, there are 3 lepton families. The charged-lepton mass matrix linking left-handed  $(e, \mu, \tau)$  to their right-handed counterparts is in general arbitrary, but may always be diagonalized by 2 unitary transformations:

$$M_l = U_L^l \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} (U_R^l)^\dagger. \quad (1)$$

Similarly, the neutrino mass matrix may also be diagonalized by 2 unitary transformations if it is Dirac:

$$M_\nu^D = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_R^\nu)^\dagger, \quad (2)$$

or by just 1 unitary transformation if it is Majorana:

$$M_\nu^M = U_L^\nu \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} (U_L^\nu)^T. \quad (3)$$

Notice that whereas the charged leptons have individual names, the neutrinos are only labeled as 1, 2, 3, waiting to be named. The observed neutrino mixing matrix is the mismatch between  $U_L^l$  and  $U_L^\nu$ , i.e.

$$U_{l\nu} = (U_L^l)^\dagger U_L^\nu \simeq \begin{pmatrix} 0.83 & 0.56 & < 0.2 \\ -0.39 & 0.59 & -0.71 \\ -0.39 & 0.59 & 0.71 \end{pmatrix} \simeq \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix}. \quad (4)$$

This approximate pattern has been dubbed tribimaximal by Harrison, Perkins, and Scott [2]. Notice that the 3 vertical columns are evocative of the mesons  $(\eta_8, \eta_1, \pi^0)$  in their  $SU(3)$  decompositions.

Historically, once the third lepton  $\tau$  was established, it was speculated by Cabibbo [3] and Wolfenstein [4] that

$$U_{\nu}^{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \quad (5)$$

where  $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$ . Note now

$$U_{\nu}^{HPS} = (U_{\nu}^{CW})^\dagger \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}. \quad (6)$$

Comparing this to Eq. (4), it tells us that if  $U_L^l$  is in fact  $U_{\nu}^{CW}$ , then  $U_{\nu}^{HPS}$  can be obtained if maximal mixing occurs in the 2 – 3 submatrix of  $M_{\nu}$ .

How can  $U_{\nu}^{HPS}$  be derived from a symmetry? The difficulty comes from the fact that any symmetry defined in the basis  $(\nu_e, \nu_{\mu}, \nu_{\tau})$  is automatically applicable to  $(e, \mu, \tau)$  in the complete Lagrangian. To do so, usually one assumes the canonical seesaw mechanism and studies the Majorana neutrino mass matrix

$$M_{\nu} = -M_{\nu}^D M_N^{-1} (M_{\nu}^D)^T \quad (7)$$

in the basis where  $M_l$  is diagonal; but the symmetry apparent in  $M_{\nu}$  (such as  $\nu_{\mu} - \nu_{\tau}$  interchange) is often incompatible with a diagonal  $M_l$  with 3 very different eigenvalues. Obviously a more sophisticated approach is needed. To obtain  $U_{\nu}^{HPS}$ , the non-Abelian discrete symmetry  $A_4$  turns out to be very useful. In this talk, I will focus mainly on this approach, but first I will discuss  $S_3$  which is the smallest non-Abelian finite group. I will also mention  $S_4$  and  $\Delta(27)$  at the end.

## 2 Permutation Symmetry $S_3$

$S_3$  is the permutation group of 3 objects, which is also the symmetry group of the equilateral triangle. It has 6 elements divided into 3 equivalence classes, with the irreducible represen-

tations  $\underline{1}$ ,  $\underline{1}'$ , and  $\underline{2}$ , and the multiplication rule  $\underline{2} \times \underline{2} = \underline{1} + \underline{1}' + \underline{2}$ . Its character table is given below.

Table 1: Character table of  $S_3$ .

class	$n$	$h$	$\chi_1$	$\chi_{1'}$	$\chi_2$
$C_1$	1	1	1	1	2
$C_2$	2	3	1	1	-1
$C_3$	3	2	1	-1	0

Let me discuss briefly 4 recent  $S_3$  models.

- Kubo, Mondragon, Mondragon, and Rodriguez-Jauregui, [5] (recently updated by Felix, Mondragon, Mondragon, and Peinado [6]): The symmetry used is actually  $S_3 \times Z_2$ , with the assignments

$$(\nu, l), l^c, N, (\phi^+, \phi^0) \sim \underline{1} + \underline{2}, \quad (8)$$

and  $v_1 = v_2$ . The  $Z_2$  symmetry serves to eliminate 4 Yukawa couplings otherwise allowed by  $S_3$ , resulting in an inverted ordering of neutrino masses with

$$\theta_{23} \simeq \pi/4, \quad \theta_{13} \simeq 0.0034, \quad m_{ee} \simeq 0.05 \text{ eV}, \quad (9)$$

where  $m_{ee}$  is the effective Majorana neutrino mass measured in neutrinoless double beta decay. This model relates  $\theta_{13}$  to the ratio  $m_e/m_\mu$ .

- Chen, Frigerio, and Ma [7]: The symmetry here is  $S_3$  only, with the assignments

$$(\nu, l) \sim \underline{1} + \underline{2}, \quad l^c \sim \underline{1} + \underline{1} + \underline{1}', \quad (\phi^+, \phi^0) \sim \underline{1} + \underline{2}, \quad (\xi^{++}, \xi^+, \xi^0) \sim \underline{2}, \quad (10)$$

and  $v_1 = v_2$  but  $u_1 \neq u_2$ . This results in a normal ordering of neutrino masses with

$$\theta_{23} \simeq \pi/4, \quad 0.008 < \theta_{13} < 0.032, \quad m_{ee} < 0.01 \text{ eV}. \quad (11)$$

This model relates  $\theta_{13}$  to  $\theta_{12}$  and the ratio  $\Delta m_{sol}^2 / \Delta m_{atm}^2$ .

- Grimus and Lavoura [8]: The symmetry is  $S_3 \times Z_2$ , with the assignments

$$\begin{aligned}
(\nu, l) &\sim (\underline{1}, +) + (\underline{2}, +), \quad l^c \sim (\underline{1}, -) + (\underline{2}, +), \quad N \sim (\underline{1}, -) + (\underline{2}, -), \\
(\phi^+, \phi^0) &\sim (\underline{1}, -) + (\underline{1}, +) + (\underline{1}', +), \quad (\chi, \chi^*) \sim (\underline{2}, +),
\end{aligned} \tag{12}$$

and  $\langle \chi \rangle^3 = \text{real}$ , resulting in a diagonal  $M_l$  and a  $\mu - \tau$  symmetric  $M_\nu$ , i.e.  $\theta_{23} = \pi/4$  and  $\theta_{13} = 0$ , whereas  $m_{ee}$  is not predicted.

- Mohapatra, Nasri, and Yu [9]: The symmetry  $S_3$  is extended to include 3  $Z_3$  transformations which do not commute with  $S_3$ , so it is not really  $S_3$ . For  $M_\nu$ , the assignments are

$$(\nu, l) \sim \underline{1} + \underline{2}, \quad N \sim \underline{1}' + \underline{2}, \quad (\phi^+, \phi^0) \sim \underline{1}, \quad (\xi^{++}, \xi^+, \xi') \sim \underline{1}, \tag{13}$$

but the extended  $S_3$  is grossly broken by  $M_N$  in a very special way, resulting then in the tribimaximal form of  $M_\nu$ . This is not what I would consider a *bona fide* derivation of  $U_{\nu}^{HPS}$ .

### 3 Tetrahedral Symmetry $A_4$

For 3 families, we should look for a group with a  $\underline{3}$  representation, the simplest of which is  $A_4$ , the group of the even permutation of 4 objects, which is also the symmetry group of the tetrahedron. The tetrahedron is one of five perfect geometric solids known to the ancient Greeks. In order to match them to the 4 elements (fire, air, earth, and water) already known, Plato invented a fifth (quintessence) as that which pervades the cosmos and presumably holds it together. In terms of symmetry, since a cube (hexahedron) may be embedded inside an octahedron and vice versa, the two must have the same group structure and are thus dual to each other. The same holds for the icosahedron and dodecahedron. The tetrahedron is self-dual. For amusement, compare this first theory of everything to today's contender, i.e.

Table 2: Perfect geometric solids in 3 dimensions.				
solid	faces	vertices	Plato	group
tetrahedron	4	4	fire	$A_4$
octahedron	8	6	air	$S_4$
cube	6	8	earth	$S_4$
icosahedron	20	12	water	$A_5$
dodecahedron	12	20	quintessence	$A_5$

string theory. (A) There are 5 consistent string theories in 10 dimensions. (B) Type I is dual to Heterotic  $SO(32)$ , Type IIA is dual to Heterotic  $E_8 \times E_8$ , and Type IIB is self-dual.

$A_4$  has 12 elements divided into 4 equivalence classes, with the irreducible representations  $\underline{1}$ ,  $\underline{1}'$ ,  $\underline{1}''$ , and  $\underline{3}$ , and the fundamental multiplication rule

$$\begin{aligned} \underline{3} \times \underline{3} = & \underline{1}(11 + 22 + 33) + \underline{1}'(11 + \omega^2 22 + \omega 33) + \underline{1}''(11 + \omega 22 + \omega^2 33) \\ & + \underline{3}(23, 31, 12) + \underline{3}(32, 13, 21). \end{aligned} \quad (14)$$

Its character table is given below, where  $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$  is exactly what we

Table 3: Character table of $A_4$ .						
class	$n$	$h$	$\chi_1$	$\chi_{1'}$	$\chi_{1''}$	$\chi_3$
$C_1$	1	1	1	1	1	3
$C_2$	4	3	1	$\omega$	$\omega^2$	0
$C_3$	4	3	1	$\omega^2$	$\omega$	0
$C_4$	3	2	1	1	1	-1

saw before in Eq. (5). Note that  $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$  is possible in  $A_4$ , i.e.  $a_1 b_2 c_3 + \text{permutations}$ , and  $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$  is possible in  $S_3$ , i.e.  $a_1 b_1 c_1 + a_2 b_2 c_2$ .

Other useful sets of finite groups are subgroups of  $SU(3)$ . The series  $\Delta(3n^2)$  has  $\Delta(3) \equiv Z_3$ ,  $\Delta(12) \equiv A_4$ ,  $\Delta(27)$ , etc. The series  $\Delta(3n^2 - 3)$  has  $\Delta(9) \equiv Z_3 \times Z_3$ ,  $\Delta(24) \equiv S_4$ , etc.

Using  $A_4$ , there are two ways to obtain  $U_{\nu}^{CW}$  as the unitary matrix which diagonalizes  $M_l$ : (I) the original proposal of Ma and Rajasekaran [10] and (II) the recent one by Ma [11].

Table 4: Representations of  $SU(3)$  and its subgroups.

$SU(3)$	$A_4$	$S_4$	$\Delta(27)$
1	1	1	$1_1$
3	3	$3'$	3
$\bar{3}$	3	$3'$	$\bar{3}$
6	$1 + 1' + 1'' + 3$	$1 + 2 + 3$	$\bar{3} + \bar{3}$
8	$1' + 1'' + 3 + 3$	$2 + 3 + 3'$	$\sum_{i=2,9} 1_i$
10	$1 + 3 + 3 + 3$	$1' + 3' + 3' + 3'$	$1_1 + \sum_{i=1,9} 1_i$

(I) Let  $(\nu_i, l_i) \sim \underline{3}$ ,  $l_i^c \sim \underline{1}, \underline{1}', \underline{1}''$ , then with  $(\phi_i^0, \phi_i^-) \sim \underline{3}$ ,

$$\begin{aligned}
 M_l &= \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 v_2 \omega & h_3 v_2 \omega^2 \\ h_1 v_3 & h_2 v_3 \omega^2 & h_3 v_3 \omega \end{pmatrix} \\
 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \sqrt{3} v,
 \end{aligned} \tag{15}$$

for  $v_1 = v_2 = v_3 = v$ .

(II) Let  $(\nu_o, l_i) \sim \underline{3}$ ,  $l_i^c \sim \underline{3}$ , then with  $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{3}$ ,

$$\begin{aligned}
 M_l &= \begin{pmatrix} h_0 v_0 & h_1 v_3 & h_2 v_2 \\ h_2 v_3 & h_0 v_0 & h_1 v_1 \\ h_1 v_2 & h_2 v_1 & h_0 v_0 \end{pmatrix} \\
 &= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix},
 \end{aligned} \tag{16}$$

for  $v_1 = v_2 = v_3$  with  $m_e = h_0 v_0 + (h_1 + h_2)v$ ,  $m_\mu = h_0 v_0 + (h_1 \omega + h_2 \omega^2)v$ , and  $m_\tau = h_0 v_0 + (h_1 \omega^2 + h_2 \omega)v$ .

In either case,  $U_{l\nu}^{CW}$  has been derived. Each allows arbitrary values of the charged-lepton

masses, and yet retains a symmetry for us to consider  $M_\nu$ . Let

$$M_\nu = \begin{pmatrix} a+b+c & f & e \\ f & a+b\omega+c\omega^2 & d \\ e & d & a+b\omega^2+c\omega \end{pmatrix} \quad (17)$$

be the Majorana neutrino mass matrix in question. Under  $A_4$ ,  $a$  comes from  $\underline{1}$ ,  $b$  from  $\underline{1}'$ ,  $c$  from  $\underline{1}''$ , and  $(d, e, f)$  from  $\underline{3}$ . Since there are 6 free parameters, this is the most general symmetric mass matrix. To proceed further, these 6 parameters must be restricted.

## 4 Selected $A_4$ Models

Using **(I)**, the first two proposed  $A_4$  models start with only  $a \neq 0$ , yielding thus 3 degenerate neutrino masses. In Ma and Rajasekaran [10], the degeneracy is broken softly by  $N_i N_j$  terms, allowing  $b, c, d, e, f$  to be nonzero. In Babu, Ma, and Valle [12], the degeneracy is broken radiatively through flavor-changing supersymmetric scalar lepton mass terms. In both cases,  $\theta_{23} \simeq \pi/4$  is predicted. In the latter, maximal CP violation in  $U_{l\nu}$  is also predicted. Consider the case  $b = c$  and  $e = f = 0$  [13], then

$$M_\nu = \begin{pmatrix} a+2b & 0 & 0 \\ 0 & a-b & d \\ 0 & d & a-b \end{pmatrix}, \quad (18)$$

which is diagonalized by

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \quad (19)$$

with eigenvalues  $a - b + d$ ,  $a + 2b$ , and  $-a + b + d$ . Comparing this with Eq. (6), we see that tribimaximal mixing has been achieved. However, since  $\underline{1}'$  and  $\underline{1}''$  are unrelated in  $A_4$ ,  $b = c$  is rather *ad hoc*. A very clever solution by Altarelli and Feruglio [14, 15] is to eliminate both, then  $b = c = 0$  naturally. This results in a normal ordering of neutrino masses with the prediction [16]

$$|m_{\nu_e}|^2 \simeq |m_{ee}|^2 + \Delta m_{atm}^2/9. \quad (20)$$



A closely related model by Babu and He [17] has  $e = f = 0$ ,  $b = c$ , and  $d^2 = 3b(b - a)$ . Here both normal and inverted ordering of neutrino masses are allowed. The technical challenge in this common approach is to break  $A_4$  spontaneously along two incompatible directions: (1,1,1) and (1,0,0). One recent proposal [18] is to add  $Z_3$  in a supersymmetric model, with singlets carrying the  $A_4$  symmetry at a high scale, and require the breaking of  $A_4$  without breaking the supersymmetry.

As for possible deviations from tribimaximal mixing, although  $b \neq c$  would allow  $U_{e3}$  to be different from zero, the assumption  $e = f = 0$  means that  $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$  remains an eigenstate. The experimental bound  $|U_{e3}| < 0.16$  then implies [13]  $0.5 < \tan^2 \theta_{12} < 0.52$ , whereas experimentally,  $\tan^2 \theta_{12} = 0.45 \pm 0.05$ .

**(III)** A third  $A_4$  scenario [19] is to have  $(\nu_i, l_i) \sim \underline{3}$ ,  $l_i^c \sim \underline{3}$ , but with  $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{1}', \underline{1}''$ . The charged-lepton mass matrix is now diagonal and  $M_\nu^{(e,\mu,\tau)} = M_\nu$  already. Using again Eq. (17) but with  $d = e = f$ ,

$$M_\nu = \begin{pmatrix} a+b+c & d & d \\ d & a+b\omega+c\omega^2 & d \\ d & d & a+b\omega^2+c\omega \end{pmatrix}. \quad (21)$$

Assume  $b = c$  and rotate to the basis  $[\nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}, (-\nu_\mu + \nu_\tau)/\sqrt{2}]$ , then

$$M_\nu = \begin{pmatrix} a+2b & \sqrt{2}d & 0 \\ \sqrt{2}d & a-b+d & 0 \\ 0 & 0 & a-b-d \end{pmatrix}, \quad (22)$$

i.e. maximal  $\nu_\mu - \nu_\tau$  mixing and  $U_{e3} = 0$ . The solar mixing angle is now given by  $\tan 2\theta_{12} = 2\sqrt{2}d/(d - 3b)$ . For  $b \ll d$ ,  $\tan 2\theta_{12} \rightarrow 2\sqrt{2}$ , i.e.  $\tan^2 \theta_{12} \rightarrow 1/2$ , but  $\Delta m_{sol}^2 \ll \Delta m_{atm}^2$  implies  $2a + b + d \rightarrow 0$ , so that  $\Delta m_{atm}^2 \rightarrow 6bd \rightarrow 0$  as well. Therefore,  $b \neq 0$  is required, and  $\tan^2 \theta_{12} \neq 1/2$ , but should be close to it, because  $b = 0$  enhances the symmetry of  $M_\nu$  from  $Z_2$  to  $S_3$ . Here  $\tan^2 \theta_{12} < 1/2$  implies inverted ordering and  $\tan^2 \theta_{12} > 1/2$  implies normal ordering.

## 5 $S_4$ and $\Delta(27)$

In the above (III) application of  $A_4$ , approximate tribimaximal mixing involves the *ad hoc* assumption  $b = c$ . This problem is overcome by using  $S_4$  in a supersymmetric seesaw model [20], yielding the result

$$M_\nu(S_4) = \begin{pmatrix} a+2b & e & e \\ e & a-b & d \\ e & d & a-b \end{pmatrix}. \quad (23)$$

Here  $b = 0$  and  $d = e$  are related limits. A more recent proposal [21] uses  $\Delta(27)$ , resulting in

$$M_\nu(\Delta(27)) = \begin{pmatrix} fa & c & b \\ c & fb & a \\ b & a & fc \end{pmatrix}. \quad (24)$$

The permutation group of 4 objects is  $S_4$ . It contains both  $S_3$  and  $A_4$ . It is also the symmetry group of the hexahedron (cube) and the octahedron. It has 24 elements divided into 5 equivalence classes, with 5 irreducible representations  $\underline{1}, \underline{1}', \underline{2}, \underline{3}, \underline{3}'$ . The fundamental multiplication rules are

$$\begin{aligned} \underline{3} \times \underline{3} &= \underline{1}(11 + 22 + 33) + \underline{2}(11 + \omega^2 22 + \omega 33, 11 + \omega 22 + \omega^2 33) \\ &+ \underline{3}(23 + 32, 31 + 13, 12 + 21) + \underline{3}'(23 - 32, 31 - 13, 12 - 21), \end{aligned} \quad (25)$$

$$\underline{3}' \times \underline{3}' = \underline{1} + \underline{2} + \underline{3}_S + \underline{3}'_A, \quad \underline{3} \times \underline{3}' = \underline{1}' + \underline{2} + \underline{3}'_S + \underline{3}_A. \quad (26)$$

Note that both  $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$  and  $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$  are possible in  $S_4$ . Let  $(\nu_i, l_i), l_i^c, N_i \sim \underline{3}$  under  $S_4$ . Assume singlet superfields  $\sigma_{1,2,3} \sim \underline{3}$  and  $\zeta_{1,2} \sim \underline{2}$ , then

$$M_N = \begin{pmatrix} M_1 & h\langle\sigma_3\rangle & h\langle\sigma_2\rangle \\ h\langle\sigma_3\rangle & M_2 & h\langle\sigma_1\rangle \\ h\langle\sigma_2\rangle & h\langle\sigma_1\rangle & M_3 \end{pmatrix}, \quad (27)$$

where  $M_1 = A + f(\langle\zeta_2\rangle + \langle\zeta_1\rangle)$ ,  $M_2 = A + f(\langle\zeta_2\rangle\omega + \langle\zeta_1\rangle\omega^2)$ , and  $M_3 = A + f(\langle\zeta_2\rangle\omega^2 + \langle\zeta_1\rangle\omega)$ .

The most general  $S_4$ -invariant superpotential of  $\sigma$  and  $\zeta$  is given by

$$\begin{aligned} W &= M(\sigma_1\sigma_1 + \sigma_2\sigma_2 + \sigma_3\sigma_3) + \lambda\sigma_1\sigma_2\sigma_3 + m\zeta_1\zeta_2 + \rho(\zeta_1\zeta_1\zeta_1 + \zeta_2\zeta_2\zeta_2) \\ &+ \kappa[(\sigma_1\sigma_1 + \sigma_2\sigma_2\omega + \sigma_3\sigma_3\omega^2)\zeta_2 + (\sigma_1\sigma_1 + \sigma_2\sigma_2\omega^2 + \sigma_3\sigma_3\omega)\zeta_1]. \end{aligned} \quad (28)$$

The resulting scalar potential has a minimum at  $V = 0$  (thus preserving supersymmetry) only if  $\langle \zeta_1 \rangle = \langle \zeta_2 \rangle$  and  $\langle \sigma_2 \rangle = \langle \sigma_3 \rangle$ , so that  $M_N$  is of the form given by Eq. (23). To obtain  $M_\nu$  of the same form,  $M_l$  should be diagonal and  $M_{\nu N}$  proportional to the identity. These are both possible with  $\phi_{1,2,3}^l \sim \underline{1} + \underline{2}$ ,  $\phi_{1,2,3}^N \sim \underline{1} + \underline{2}$ , but with zero vacuum expectation value for  $\phi_{2,3}^N$ .

$\Delta(27)$  has 27 elements divided into 11 equivalence classes. There are 9 one-dimensional irreducible representations  $\underline{1}_i$  and 2 three-dimensional ones  $\underline{3}, \bar{\underline{3}}$ , with the multiplication rules

$$\underline{3} \times \underline{3} = \bar{\underline{3}} + \bar{\underline{3}} + \bar{\underline{3}}, \quad \underline{3} \times \bar{\underline{3}} = \sum_{i=1,9} \underline{1}_i. \quad (29)$$

For the product  $\underline{3} \times \underline{3} \times \underline{3}$ , there are 3 invariants:  $123 + 231 + 312 - 213 - 321 - 132$  which is invariant under  $SU(3)$ ,  $123 + 231 + 312 + 213 + 321 + 132$  which is also invariant under  $A_4$ , and  $111 + 222 + 333$ . Let  $(\nu_i, l_i) \sim \underline{3}$ ,  $l_i^c \sim \bar{\underline{3}}$ ,  $(\phi_i^0, \phi_i^-) \sim \underline{1}_{1,2,3}$ ,  $(\xi_i^{++}, \xi_i^+, \xi_i^0) \sim \underline{3}$ , then Eq. (24) is obtained. Again let  $b = c$ , then two solutions for example are  $f = 1.1046$  and  $f = -0.5248$ , for both of which  $\tan^2 \theta_{12} = 0.45$  and  $m_{ee} = 0.05$  eV.

## 6 Conclusion

With the application of the non-Abelian discrete symmetry  $A_4$ , a plausible theoretical understanding of the tribimaximal form of the neutrino mixing matrix has been achieved. Other symmetries such as  $S_4$  and  $\Delta(27)$  are beginning to be studied. They share some of the properties of  $A_4$  and may help to extend our understanding of possible discrete family symmetries, with eventual links to grand unification [22].

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